1. *Show that the set of polynomials with integer coefficients:*

$$\mathcal{P} = \{p : p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \quad n \in \mathbb{N}\bigcup\{0\}, a_j \in \mathbb{Z}, 0 \leq j \leq n, a_n \neq 0\}$$

is countable.

Solution: For each $n \in \mathbb{N}\bigcup\{0\}$, let $\mathcal{P}_n$ denote the set of polynomials degree n with integer coefficients. i.e.,

$$\mathcal{P}_n = \{p : p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, a_j \in \mathbb{Z}, 0 \leq j \leq n, a_n \neq 0\}$$

Then $\mathcal{P}_n$ is bijective to the set $A = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Since finite product of countable set is countable, A is countable and thus $\mathcal{P}_n$ is countable for each $n \in \mathbb{N}\bigcup\{0\}$. As countable union of countable sets, $\mathcal{P} = \bigcup\limits_{n=0}^{\infty} \mathcal{P}_n$ is countable. □

2. *Prove that the set of natural numbers is not bounded above.*

Solution: Assume by the way of contradiction that $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is non-empty set, it follows from the l.u.b axiom that $\sup(\mathbb{N})$ exists. Thus, there must be $m \in \mathbb{N}$ such that $\sup(\mathbb{N}) - 1 < m$. This implies that $\sup(\mathbb{N}) < m + 1$, where $m + 1 \in \mathbb{N}$. This is a contradiction. Therefore, $\mathbb{N}$ is not bounded above. □

3. *Let $I_n = [a_n, b_n]$ be a nested family of closed intervals, that is, $a_n, b_n \in \mathbb{R}, a_n < b_n$, with $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for every n. Assume that $\lim_{n\to\infty}(b_n - a_n) = 0$. Suppose $\{x_n\}_{n\geq1}$ is a sequence of real numbers, where $x_n \in I_n$ for every n. Show that $\{x_n\}_{n\geq1}$ is convergent.*

Solution: By Cantor's intersection theorem (See Theorem 3.10, Priniciples of Mathematical Analysis by Walter Rudin), $\bigcap\limits_{n=0}^{\infty} I_n$ is exactly a singleton set, $\{x\}$ (say). Thus $x \in I_n$ for all n. Consequently, we have

$$|x_n - x| \leq \operatorname{diam}(I_n) = b_n - a_n.$$

Since $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, $\{x_n\}_{n\geq1}$ is convergent to x. □

4. *Find* $\limsup$ *and* $\liminf$ *of following sequences of real numbers:*

   (i) $\{a_n\}_{n\geq1}$ where $a_n = \frac{1}{n+1} - \frac{1}{n}$ for $n \in \mathbb{N}$;

   Solution: Since

$$a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} \to 0 \text{ as } n \to \infty,$$

   $\limsup a_n = \liminf a_n = \lim\limits_{n\to\infty} a_n = 0$.

   (ii) $\{b_n\}_{n\geq1}$ where

$$b_n = \begin{cases} 3 & \text{if } n = 3k-2, k \in \mathbb{N} \\ 5 + \frac{10}{n} & \text{if } n = 3k-1, k \in \mathbb{N} \\ 6 & \text{if } n = 3k, k \in \mathbb{N} \end{cases}$$

Solution: All the subsequential limits of b_n's are $3, 5$ and 6. Hence, $\limsup a_n = 6$ and $\liminf a_n = 3$.

$\square$

5. *Show that a sequence $\{a_n\}_{n\geq 1}$ of real numbers is convergent if and only if it is Cauchy.*

Solution: See part (a) and part (c) of Theorem 3.11 from the book on Priniciples of Mathematical Analysis by Walter Rudin. $\square$

6. *Find the set of cluster points of following subsets of* $\mathbb{R}$:

(i) $A = \{2 + \frac{(-1)^n}{7n} : n \in \mathbb{N}\}$;

Solution: As $\frac{(-1)^n}{7n} \to 0$ as $n \to \infty$, we get $2 + \frac{(-1)^n}{7n} \to 2$ as $n \to \infty$. Thus, cluster points of the set A is the singleton set $\{2\}$.

(ii) $B = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$.

Solution: For each fixed $m \in \mathbb{N}$, $\frac{1}{m} + \frac{1}{n} \to \frac{1}{m}$ as $n \to \infty$ and $\frac{1}{m} + \frac{1}{n} \to 0$ when m and n approches ∞ together. Hence, cluster points of the set B is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

$\square$

7. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying*

$$|h(x) - h(y)| \leq 10|x - y|, \qquad \forall x, y \in \mathbb{R}.$$

Show that h is continuous.

Solution: Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/10$. If $|x - y| < \delta$, then

$$|h(x) - h(y)| \leq 10|x - y| < 10\delta = \epsilon.$$

Hence, h is continuous at every point of $\mathbb{R}$.

$\square$